Measurements of Microwave Conductivity and Dielectric Constant by the Cavity Perturbation Method and Their Errors

SHUH-HAN CHAO

Abstract —**The theory and technique of the cavity perturbation method for measuring the conductivity and dielectric constant of materials are reviewed. An analytical formula for calculating the errors of the conductivity and dielectric constant caused by the measured error in the resonant frequency and quality factor are derived. This formula can be used for both rectangular and cylindrical cavities. The results of measurements on silicon samples are presented to illustrate this analysis.**

I. Introduction

CAVITY PERTURBATION methods have been widely used to measure the conductivity σ and the dielectric constant ε of materials at microwave frequencies. These measurements are performed by inserting a small, appropriately shaped sample into a cavity and determining the properties of the sample from the resultant change in the resonant frequency and Q-factor.

The earliest treatment of the cavity perturbation theory was given by Bethe and Schwinger [1]; they considered the cases that the perturbation causes 1) by the insertion of a small dielectric sample into cavity and 2) by a small deformation of the boundary surface of the cavity. Later, Casimar [2] extended the cavity perturbation theory to include the determination of the magnetic property of a small sphere.

The basic assumption of the cavity perturbation is that the change in the overall geometrical configuration of the electromagnetic fields upon introduction of the sample must be small. Experimentally, this means that the percentage change in the real part of the resonant frequency must be small [3]. Based on this assumption, a detailed derivation of the perturbation formula for the frequency shift upon introduction of a sample into a resonant cavity was given by Waldron [4].

The first application of cavity perturbation techniques for the measurement of σ and ε was developed by Birbaum and Franeau [5]. In their experimental arrangement, a small cylindrical sample is placed in a rectangular cavity, operating in the TE_{106} mode. An additional assumption that "the electric field in the perturbing sample is equal to the electric field of the empty cavity" was made in their calculation. The results of loss tangent and dielectric constant for some low-loss liquids and solids were obtained.

A practical application of the TM_{010} mode cylindrical cavity was reported by Nakamura and Furnichi [6], who performed measurements upon a cylindrical shaped $BaTiO_3$ single crystal placed along the axis of the cavity. This technique has also been used by Barker *et al.* [7] for the measurement of the ionic conductor β-alumina. Generally, in this technique, the length of the sample is equal to the height of the cavity so that both ends of the sample are in contact with the cavity walls. However, since the resonant condition of a TM_{010} cavity is not dependent upon the length, it can be made "flat" and a short sample can be used. Later, Parkash *et al.* [8] considered the case in which the sample did not contact the cavity wall. They introduced a set of formulas for calculating σ and ε based on the assumption that the sample acts like a dipole with an effective depolarizing factor Ne. Their formulas have been found to be adequate in yielding consistent results when applied to the sample of a length less than the height of cavity.

In the cavity perturbation method, small holes can be drilled in the cavity walls and the sample can then be inserted into the sample holder. By using this technique, the cavity need not be taken apart for placing the sample, and errors due to the misalignment of the cavity may be reduced. This technique was used by Labuda and Lecrew [9] and by Buranov and Shchegoler [10].

A different type of cavity, the reentrant cavity, has also been used by some investigators for measurements of σ and ε [11], [12]. This type of cavity has a strong electric field in certain regions, which results in a greater interaction between the sample and the field. Hence, the reentrant cavity is a good selection for the measurement of low-conductivity materials. Recently, a tunable reentrant cavity has been presented by Kaczkowski and Milewski [13]. Since the dimensions of the cavity are adjustable, samples with various lengths can be inserted and the resonant condition can be obtained from the adjustments of the cavity length. A relatively wide range of materials ($\varepsilon_r = 2-300$, $\tan\delta = 10^{-5}$ to 10^{-1}) were measured in their work.

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The author is with the Materials R & D Center, Chung Shan Institute of Science and Technology, Lung-Tan, Taiwan 325, Republic of China.

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A paper presented by Champlin and Krongard [14] treated the problem of determining σ and ε of a small sphere ($\sim$1-mm radius for X-band measurements) of arbitrary conductivity. In their paper, sphere semiconductor samples placed at the location of the electric-field maximum ($E_{\max}$) were measured by using a rectangular cavity operating in the TE_{103} mode. This method was further discussed by Mansingh and Parkash [15], who introduced a simplifying approximation to allow the conductivity and the dielectric constant to be obtained in close form, eliminating the need for computer iteration.

For relatively high-conductivity materials where the skin depth δ is smaller than the sample dimensions, the eddy current loss measuring technique presented by Linhart *et al.* [16] may be useful. In this method, the sample is placed at the location of a magnetic-field maximum ($H_{\max}$). Boundary conditions of the H-field require the induction of currents close to the surface of the sample, thus giving rise to an energy loss. This energy loss will appear as a change in the Q-factor, which can be used to determine the conductivity of the sample.

For measurements of any size sample of arbitrary σ and ε, an exact solution of the fields in the interior of the perturbing sample is necessary. Brodwin and Parsons [17] treated this problem by considering a plane wave diffracted by a spherical sample with arbitrary size. An exact solution of the electric field in the interior of the sample is given. This solution is valid for any value of sample radius, conductivity, and dielectric constant as long as the percentage change in the real part of the resonant frequency is small. Recently, Gebhardt *et al.* [18] applied this exact solution to the measurements of σ and ε of the ionic conductor $\alpha-\mathrm{AgI}$ ($\sigma = 1/\Omega\cdot\mathrm{cm}$) with relatively large sample size (diameter ranges from 1.4 to 7.0 mm). In their experimental arrangement, an open resonator (Fabry–Perot cavity) operating in a TEM_{00q} mode was used.

The errors of measurements of σ and ε by the cavity perturbation method have been reported in only a few articles. In Rzepecka's paper [19], cylindrical samples were measured by using a rectangular cavity. The electric field in the perturbed cavity was assumed to be equal to the electric field in the empty cavity. The errors of σ and ε caused by measured errors in the resonant frequency and Q-factor were considered. In Kaczkowski and Milewski's [13] paper (mentioned previously), cylindrical samples were measured by using a tunable reentrant cavity. The errors of σ and ε caused by the errors in the measurements of dimensions of the cavity, the resonant frequency and Q-factor, and the dimensions of the samples were examined.

In this paper, an error analysis of the cavity perturbation method will be performed. This analysis will cover a relatively wide range of conductivity, 10^{-4} to $1.0/\Omega\cdot\mathrm{cm}$.[1] The analytical formula for calculating the errors of σ and ε caused by the measured error in the resonant frequency and Q-factor will be derived. This formula can be used for both rectangular and cylindrical cavities. We commence with a review of the theory of the cavity perturbation method. Next, the errors of σ and ε are calculated. Finally, the results of measurements on silicon samples with various dimensions are presented to illustrate this analysis.

[1] In this paper, all units for conductivity are in $(\Omega\cdot\mathrm{cm})^{-1}$.

II. Measurement Theory

A sample of materials inserted in a resonant cavity will cause the complex frequency to change by an amount. The frequency shift may be written as [4]

$$\frac{\delta\tilde{\omega}}{\tilde{\omega}} = -\frac{(\tilde{\varepsilon}_r - 1)\varepsilon_0 \int_{Vs} E\cdot E_0^*\,dV + (\tilde{\mu}_r - 1)\mu_0 \int_{Vs} H\cdot H_0^*\,dV}{\int_{Vc}\left(D_0\cdot E_0^* + B_0\cdot H_0^*\right)dV} \tag{1}$$

where $\delta\tilde{\omega}/\tilde{\omega}$ is the complex resonant frequency shift; B_0, H_0, D_0, and E_0 are the cavity fields which are assumed to have the same configuration as the unperturbed cavity mode; E and H are the fields in the interior of the perturbing sample. $\tilde{\varepsilon}_r = \varepsilon_r - j\sigma/\omega\varepsilon_0$ is the complex relative permittivity and $\tilde{\mu}_r$ is the complex relative permeability. V_s and V_c are the volumes of the sample and the cavity, respectively. The limitation on the validity of (2) is that $\delta\tilde{\omega} \ll \tilde{\omega}$.

For a small nonmagnetic sample ($\mu_r = 1$) placed at the electric-field maximum, the electric field applied to the sample can be assumed uniform, and (1) can be simplified as

$$\frac{\delta\tilde{\omega}}{\tilde{\omega}} = -\frac{(\tilde{\varepsilon}_r - 1)\varepsilon_0 \int_{Vs} E\cdot E_{0\max}^*\,dV}{2\int_{Vc}\varepsilon_0|E_o|^2\,dV} = -\frac{P\cdot E_{0\max}^*}{2\int_{Vc}\varepsilon_0|E_0|^2\,dV} \tag{2}$$

where P is the total induced electric dipole moment.

Assume that the Q-factor of the perturbed cavity is very high; the complex frequency shift $\delta\tilde{\omega}/\tilde{\omega}$ can be separated into real and imaginary parts as

$$\frac{\delta\tilde{\omega}}{\tilde{\omega}} = \frac{\delta f_0}{f_0} + j\delta\left(\frac{1}{2Q_0}\right) = -\frac{P\cdot E_{0\max}^*}{2\int_{V_c}\varepsilon_0|E_0|^2\,dV} \tag{3}$$

where

$$\left(\frac{\delta f_0}{f_0}\right) = \left(\frac{f_{0e} - f_{0s}}{f_{0s}}\right) \doteq \left(\frac{f_{0e} - f_{0s}}{f_{0e}}\right)$$

and

$$\delta\left(\frac{1}{2Q_0}\right) = \tfrac{1}{2}\left(\frac{1}{Q_{0s}} - \frac{1}{Q_{0e}}\right).$$

Here, f_{0s} and Q_{0s} are the resonant frequency and Q-factor of the unloaded cavity with the sample inserted, and f_{0e} and Q_{0e} are the corresponding quantities for the unloaded empty cavity. Once the quantities f_{0s}, f_{0e}, Q_{0s}, and Q_{0e} are known, the conductivity and the dielectric constant can be determined by (3).

In order to determine σ and ε, the solution of the total induced electric moment for the extended quasistatic approximation is given by [17]

$$P=\frac{3\varepsilon_0 E_{0\,\max} V_s}{2}\left[\frac{2\tilde{\varepsilon}_r j_1(N\rho)-[N\rho j_1(N\rho)]'}{\tilde{\varepsilon}_r j_1(N\rho)+N\rho j_1(N\rho)]'}\right] \tag{4}$$

where $j_1(N\rho)$ is the spherical Bessel function of order one, $\rho=(\omega\sqrt{\mu_0\varepsilon_0})a$, a is the sample radius, and $N=(\tilde{\mu}_r\tilde{\varepsilon}_r)^{1/2}$. Equation (4) can be rearranged to give

$$P=3\varepsilon_0 E_{0\,\max} V_s\left[\frac{\tilde{\varepsilon}_r\dfrac{2j_1(N\rho)}{[N\rho j_1(N\rho)]'}-1}{\tilde{\varepsilon}_r\dfrac{2j_1(N\rho)}{[N\rho j_1(N\rho)]'}+2}\right]$$

$$=3\varepsilon_0 E_{0\,\max} V_s\left[\frac{\tilde{\varepsilon}_r g(N\rho)-1}{\tilde{\varepsilon}_r g(N\rho)+2}\right] \tag{5}$$

where

$$g(N\rho)=\frac{2j_1(N\rho)}{[N\rho j_1(N\rho)]'}$$

$$=-2\left[\frac{N\rho\cos(N\rho)-\sin(N\rho)}{N\rho\cos(N\rho)-(1-N^2\rho^2)\sin(N\rho)}\right]. \tag{6}$$

Substitution of P into (3) yields

$$\frac{\delta f_0}{f_0}+j\delta\left(\frac{1}{2Q_0}\right)=-\frac{3}{2}\left(\frac{1}{C_c}\right)\left(\frac{V_s}{V_c}\right)\left[\frac{\tilde{\varepsilon}_r g(N\rho)-1}{\tilde{\varepsilon}_r g(N\rho)+2}\right] \tag{7}$$

where

$$C_c=\frac{1}{V_c}\int_{Vc}\frac{|E_0|^2}{|E_{0\,\max}|^2}\,dV.$$

For a rectangular cavity operating in the TE_{10n} mode, E_0 is given by

$$E_0=E_{0\,\max}\sin\frac{\pi x}{A}\sin\frac{n\pi z}{D}$$

where A and D are the width and length of the cavity. The parameter C_c can be determined by

$$C_c=\frac{1}{V_c}\int_{Vc}\frac{|E_{0\,\max}\sin\frac{\pi x}{A}\sin\frac{n\pi z}{D}|^2}{|E_{0\,\max}|^2}\,dV=\frac{1}{4}.$$

Letting

$$M=\frac{3}{2}\left(\frac{1}{C_c}\right)\left(\frac{V_s}{V_c}\right)$$

(7) can be rewritten as

$$\frac{\delta f_0}{f_0}+j\delta\left(\frac{1}{2Q_0}\right)=-M\left[\frac{\tilde{\varepsilon}_r g(N\rho)-1}{\tilde{\varepsilon}_r g(N\rho)+2}\right]. \tag{8}$$

To determine $\tilde{\varepsilon}_r$, let

$$\tilde{\varepsilon}_r g(N\rho)=u-jv. \tag{9}$$

The relationship between $(\delta f_0/f_0)$, $\delta(1/2Q_0)$, u, and v can be given by [14]

$$1+\frac{1}{M}\frac{\delta f_0}{f_0}=\frac{3(u+2)}{(u+2)^2+v^2} \tag{10}$$

$$\frac{1}{M}\delta\left(\frac{1}{2Q_0}\right)=\frac{3v}{(u+2)^2+v^2} \tag{11}$$

and

$$u=\frac{3\left\{1+\frac{1}{M}\left(\frac{\delta f_0}{f_0}\right)\right\}}{\left\{1+\frac{1}{M}\left(\frac{\delta f_0}{f_0}\right)\right\}^2+\left\{\frac{1}{M}\delta\left(\frac{1}{2Q_0}\right)\right\}^2}-2 \tag{12}$$

$$v=\frac{3\left\{\frac{1}{M}\delta\left(\frac{1}{2Q_0}\right)\right\}}{\left\{1+\frac{1}{M}\left(\frac{\delta f_0}{f_0}\right)\right\}^2+\left\{\frac{1}{M}\delta\left(\frac{1}{2Q_0}\right)\right\}^2}. \tag{13}$$

Both u and v can be calculated from (12) and (13) after the resonant frequency shift and Q-factor change are known. The conductivity and the dielectric constant can then be determined by

$$\tilde{\varepsilon}_r=\varepsilon_r-j\frac{\sigma}{\omega\varepsilon_0}=(u-jv)/g(N\rho). \tag{14}$$

For highly conductive materials, the Q-factor change is small due to the skin depth effect. Under this condition, samples can be placed in the magnetic-field maximum instead of the electric-field maximum to yield a larger Q-factor change. Assume that a spherical sample is placed at the magnetic-field maximum; a magnetic moment is obtained as

$$M=3\mu_0 H_{0\,\max} V_s\left[\frac{\tilde{\mu}_r g(N\rho)-1}{\tilde{\mu}_r g(N\rho)+2}\right]$$

$$=3\mu_0 H_{0\,\max} V_s\left[\frac{g(N\rho)-1}{g(N\rho)+2}\right]. \tag{15}$$

The Q-factor change due to the insertion of the sample is given by

$$\frac{1}{M}\delta\left(\frac{1}{2Q_0}\right)=\frac{-3\,\mathrm{Im}[g(N\rho)]}{\{\mathrm{Re}[g(N\rho)]+2\}^2+\{\mathrm{Im}[g(N\rho)]\}^2} \tag{16}$$

where $\mathrm{Re}[g(N\rho)]$ and $\mathrm{Im}[g(N\rho)]$ are the real and imaginary parts of $g(N\rho)$, respectively. For highly conductive materials ($\sigma/\omega\varepsilon_0 \gg \varepsilon_r$ and $|N\rho|\gg 1$), the Q-factor change can be related directly to conductivity by

$$\frac{1}{M}\delta\left(\frac{1}{2Q_0}\right)=\frac{3}{2\sqrt{2}\,\omega}\frac{1}{(\mu_0\varepsilon_0)^{\frac{1}{2}}a}\left(\frac{\sigma}{\omega\varepsilon_0}\right)^{-\frac{1}{2}}. \tag{17}$$

In this paper, measurements will be reported under the condition that the sample is placed at the electric-field maximum. A numerical iterative method is needed to obtain σ and ε by solving (14). A Fortran program which solves this equation using the Newton–Raphson method is available from the author.

III. Error Analysis

The errors in measurements of σ and ε by the cavity perturbation method depend not only upon the accuracies of the measurements of the resonant frequency and Q-factor but also upon the validity of the approximations made in the determination of the electric field in the interior of the perturbing sample. This latter point is not considered in our analysis. In real experiments, small samples are chosen to reduce the errors of the perturbation approximation. However, the sample size should not be too small, otherwise the changes of the resonant frequency and the Q-factor due to the insertion of the sample are small and the errors of σ and ε caused by the measured error of the resonant frequency and Q-factor are large. It implies that for a given material, due to the conflicting requirements of small size for small perturbation error and large size for small percentage errors in δf_0 and δQ_0, an optimum sample size for minimum errors can be found.

In this section, an analysis of the error of the conductivity and the dielectric constant will be performed. In this analysis, small samples are considered ($a = 0.5$ to 1.5 mm) and attention is given to the errors caused by the measured errors in the resonant frequency and the Q-factor. It is done by the following procedure.

Step 1. The resonant frequency shift δf_0 and the Q-factor change $\delta(1/Q_0)$ as functions of the conductivity, dielectric constant, and sample radius will be calculated. The reason for doing these calculations is to understand the effect of the choice of σ, ε, and a to the results of δf_0 and $\delta(1/Q_0)$.

Step 2. The analytical forms of the errors of $\Delta\sigma/\sigma$ and $\Delta\varepsilon/\varepsilon$ as functions of $\Delta f_0/f_0$ and $\Delta Q_0/Q_0$ will be derived, where Δf_0 and ΔQ_0 are the measured errors of the resonant frequency and Q-factor. These analytical forms allow: 1) the calculation of errors of $\Delta\sigma/\sigma$ and $\Delta\varepsilon/\varepsilon$ after $\Delta f_0/f_0$ and $\Delta Q_0/Q_0$ are known and 2) the determination of the required values of $\Delta f_0/f_0$ and $\Delta Q_0/Q_0$ after the limits of $\Delta\sigma/\sigma$ and $\Delta\varepsilon/\varepsilon$ are set. For example, from the theory of the total derivative

$$\frac{\Delta\sigma}{\sigma} = a_{11}\left(\frac{\Delta f_0}{f_0}\right) + a_{12}\left(\frac{\Delta Q_0}{Q_0}\right)$$

$$\frac{\Delta\varepsilon}{\varepsilon} = a_{21}\left(\frac{\Delta f_0}{f_0}\right) + a_{22}\left(\frac{\Delta Q_0}{Q_0}\right)$$

and we assume that a_{11}, a_{12}, a_{21}, and a_{22} are constants. If $\Delta f_0/f_0$ and $\Delta Q_0/Q_0$ are known then $\Delta\sigma/\sigma$ and $\Delta\varepsilon/\varepsilon$ can be calculated, or if $\Delta\sigma/\sigma$ and $\Delta\varepsilon/\varepsilon$ are set, the requirements of $\Delta f_0/f_0$ and $\Delta Q_0/Q_0$ can then be determined.

First, the results of Step 1 are described. In Figs. 1 and 2, the resonant frequency shift δf_0 and the Q-factor change $\delta(1/Q_0)$ as functions of the sample radius for $\varepsilon = 10$ and $\sigma = 10^{-4}$ to 1.0 are plotted. Both δf_0 and $\delta(1/Q_0)$ are calculated from (10) and (11) by assuming a rectangular cavity operation in the TE_{103} mode with $Q_{0e} = 3000$, $f_{0e} =$ 10 GHz, and the sample is placed at the electric-field maximum. Figs. 3 and 4 give δf_0 and $\delta(1/Q_0)$ as functions of the conductivity where the sample radius is assumed constant and equal to 1 mm, and the relative dielectric constant is chosen from 10 to 80.

For comparison, the Q-factor change of a $a = 1$-mm sample placed at the magnetic-field maximum is calculated (16) as a function of conductivity and plotted together with the result of the case where the sample is placed at the electric-field maximum, as shown in Fig. 5. Note that, in

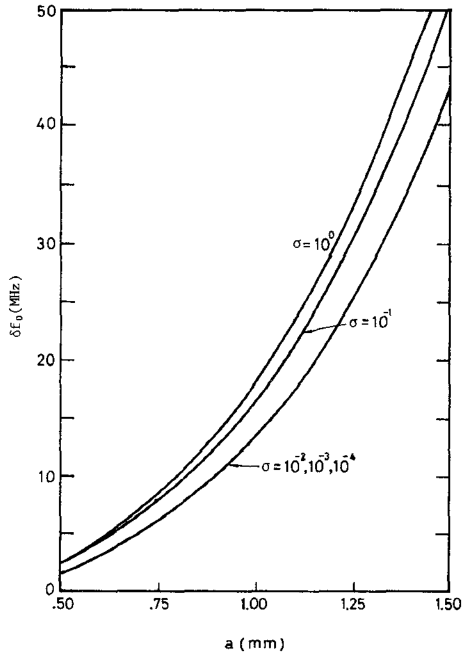

Fig. 1. Resonant frequency shift as a function of sample radius for $\sigma = 10^{-4}$ to $1.0/\Omega\cdot$cm and $\varepsilon_r = 10$. (TE_{103} rectangular cavity, $Q_{0e} = 3000$, $f_{0e} = 10$ GHz, and sample placed at E_{max}.)

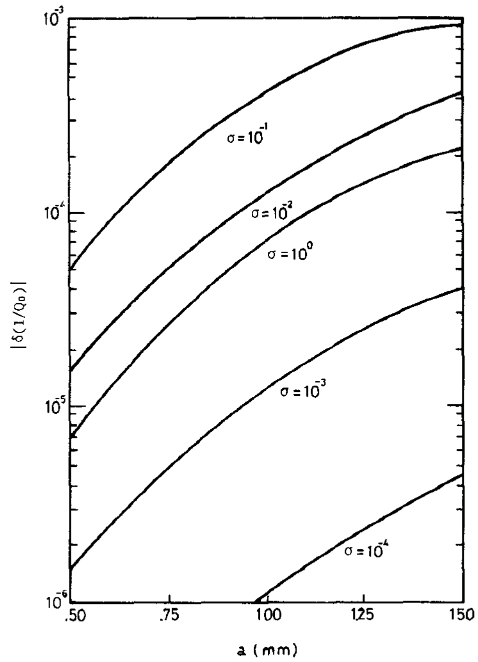

Fig. 2. Unloaded Q-factor change as a function of sample radius for $\sigma = 10^{-4}$ to $1.0/\Omega\cdot$cm and $\varepsilon_r = 10$. (Same conditions as those in Fig. 1.)

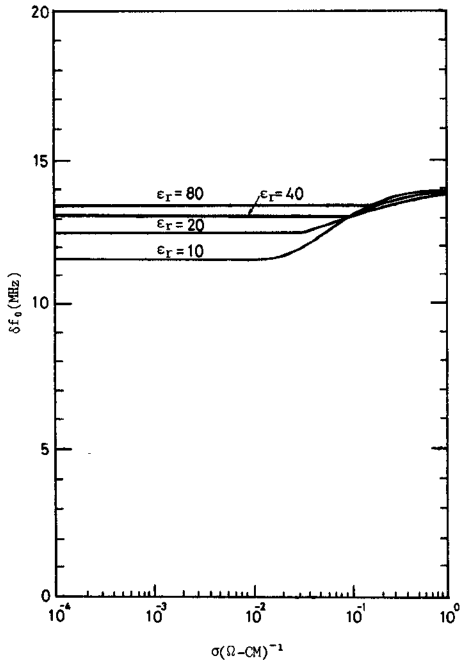

Fig. 3. Resonant frequency shift as a function of conductivity for $\varepsilon_r = 10$ to 80 and $a = 1$ mm. (Same conditions as those in Fig. 1.)

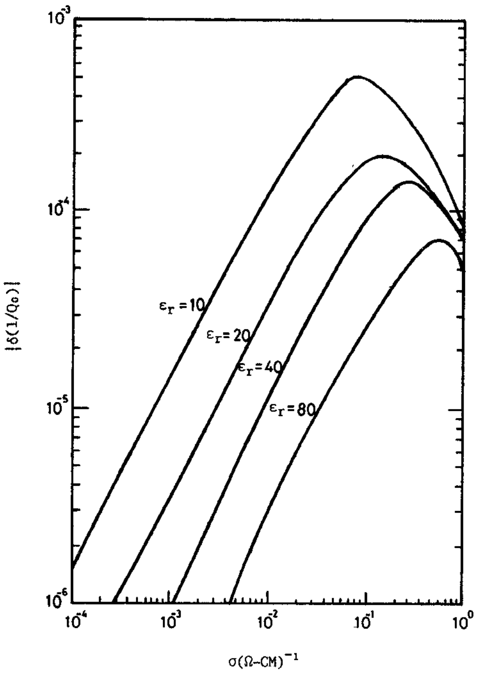

Fig. 4. Unloaded Q-factor change as a function of conductivity for $\varepsilon_r = 10$ to 80 and $a = 1$ mm. (Same conditions as those in Fig. 1.)

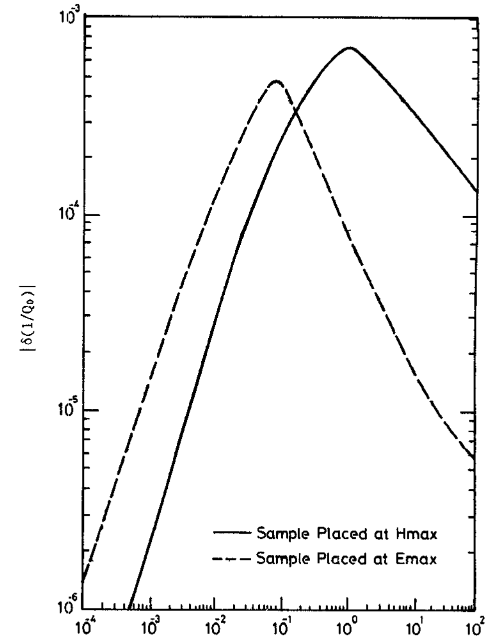

Fig. 5. Comparison of the results of unloaded Q-factor change for a sample placed at $E_{\max}$ with the results for the same sample placed at $H_{\max}$. (TE_{103} rectangular cavity, $Q_{0e} = 3000$, $f_{0e} = 10$ GHz, $\varepsilon_r = 10$, and $a = 1$ mm.)

this figure, the conductivity range is from 10^{-4} to 10^{2}. The reason for extending the upper limit from 1.0 to 10^{2} is to show the entire trend of the Q-factor change of the case where the sample is placed at $H_{\max}$.

From these results, the following conclusions are inferred.

1) For a given material, both the resonant frequency δf_0 and the Q-factor change $\delta(1/Q_0)$ increase with increasing sample radius ($a = 0.5$ to 1.5 mm), as shown in Figs. 1 and 2.

2) For low-conductivity materials, δf_0 is relatively independent of the conductivity (Figs. 1 and 3). The Q-factor change $\delta(1/Q_0)$ versus the conductivity is approximately linear in a log/log plot (Fig. 4), for the low-conductivity range and $\delta(1/Q_0) \leq 10^{-5}$ for $\sigma \leq 10^{-3}$ and $a = 1$ mm.

3) For medium-conductivity materials, 10^{-2} to 10^{-1}, the increase of conductivity affects both δf_0 and $\delta(1/Q_0)$. δf_0 of Fig. 3 is increased slightly as the conductivity increases in this range. $\delta(1/Q_0)$ exhibits a maximum value at the position where $\sigma/\omega\varepsilon \doteq 1$, as shown in Fig. 4. Further calculations show that the position of the maximum value is unaffected by the radius of the sample.

4) For high-conductivity materials, both δf_0 and $\delta(1/Q_0)$ are relatively independent of the dielectric constant, as shown in Figs. 3 and 4. This reveals that the dielectric constant is difficult to measure using the cavity perturbation method for a sample with high conductivity. The Maxwell's equation also shows that, as the conductivity increases, the conduction current increases and can become so large compared to the displacement current that the displacement current is no longer physically observable, and thus the dielectric constant cannot be measured.

5) For high-conductivity materials, $\sigma/\omega\varepsilon \geq 1$, $\delta(1/Q_0)$ decreases with increasing conductivity, as shown in Fig. 4. Under this condition, a sample placed at the magnetic-field maximum may produce a larger Q-factor change. Fig. 5 shows that for $\sigma > 2\times10^{-1}$, a sample placed at the magnetic-field maximum causes a larger Q-factor change than the same sample placed at the electric-field maximum.

6) Note that $\delta(1/Q_0)$ of the sample placed at $H_{\max}$, shown in Fig. 5 by a solid line, is calculated by (16). For highly conductive materials, this equation can be simplified

to (17). Further calculations show that for a $\sigma=10$ and $\varepsilon_r=10$ sample with $a=1$ mm, $\delta(1/Q_0)$ calculated from (17) has a 8.5-percent error with respect to the result calculated from (16). When σ increases to 10^2, the difference between the Q-factors calculated from (16) and (17) is decreased to 4 percent. Thus, for highly conductive materials, the conductivity can be determined from the Q-factor change by using the simple closed form (17).

7) It should be pointed out that in these calculations (Figs. 1–5), we assumed that the field applied to the sample was uniform (but the interior field need not be uniform due to the skin depth effects). However, when the sample is made larger, the applied field may not be uniform and may introduce error. To account for this, the exact solution [17], an infinite series, should be used and higher order terms should be taken into consideration.

We now return to Step 2 of this procedure. Generally, the maximum relative error of a measurement of a function of n variables expressed as $R(x_1, x_2, \cdots, x_n)$ is given by [20]

$$\frac{\Delta R}{R}=\left|\frac{x_1}{R}\left(\frac{\partial R}{\partial x_1}\cdot\frac{\Delta x_1}{x_1}\right)\right|+\left|\frac{x_2}{R}\left(\frac{\partial R}{\partial x_2}\cdot\frac{\Delta x_2}{x_2}\right)\right| \cdots+\left|\frac{x_n}{R}\left(\frac{\partial R}{\partial x_n}\cdot\frac{\Delta x_n}{x_n}\right)\right|$$

where $\Delta x_1, \Delta x_2,, \ldots, \Delta x_n$ are the errors in $x_1, x_2,, \ldots, x_n$, respectively.

In the specific case of measurements of σ and ε by the cavity perturbation method, in order to calculate the errors $\Delta\sigma/\sigma$ and $\Delta\varepsilon/\varepsilon$, it is necessary to know the partial derivatives $\partial\sigma/\partial f_0$, $\partial\sigma/\partial Q_0$, $\partial\varepsilon/\partial f_0$, and $\partial\varepsilon/\partial Q_0$, as well as assume certain errors in a selected measurement system. These partial derivatives are obtained by using (13)–(15), and the errors $\Delta\sigma/\sigma$ and $\Delta\varepsilon/\varepsilon$ are given by

$$\frac{\Delta\sigma}{\sigma}=a_{11}\left(\frac{\Delta f_0}{f_0}\right)+a_{12}\left(\frac{\Delta Q_0}{Q_0}\right)$$
$$\frac{\Delta\varepsilon}{\varepsilon}=a_{21}\left(\frac{\Delta f_0}{f_0}\right)+a_{22}\left(\frac{\Delta Q_0}{Q_0}\right) \tag{18}$$

where

$$a_{11}=\left(\frac{\omega\varepsilon_0}{\sigma}\right)\frac{2}{M}\left[\left|\frac{6Axy}{(x^2+y^2)^2}\right|+\left|\frac{3B(x^2-y^2)}{(x^2+y^2)^2}\right|\right]$$

$$a_{12}=\left(\frac{\omega\varepsilon_0}{\sigma}\right)\frac{1}{2M}\left(\frac{1}{Q_{0s}}+\frac{1}{Q_{0e}}\right)\cdot\left[\left|\frac{3A(x^2-y^2)}{(x^2+y^2)^2}\right|+\left|\frac{6Bxy}{(x^2+y^2)^2}\right|\right]$$

$$a_{21}=\left(\frac{1}{\varepsilon_r}\right)\frac{2}{M}\left[\left|\frac{3A(x^2-y^2)}{(x^2+y^2)^2}\right|+\left|\frac{6Bxy}{(x^2+y^2)^2}\right|\right]$$

$$a_{22}=\left(\frac{1}{\varepsilon_r}\right)\frac{1}{2M}\left(\frac{1}{Q_{0s}}+\frac{1}{Q_{0e}}\right)\cdot\left[\left|\frac{6Axy}{(x^2+y^2)^2}\right|+\left|\frac{3B(x^2-y^2)}{(x^2+y^2)^2}\right|\right]$$

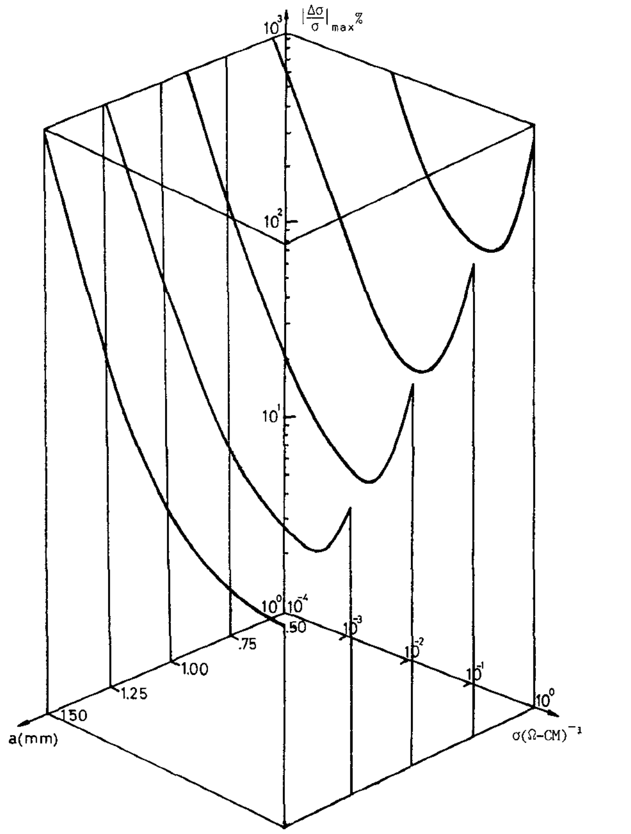

Fig. 6. The maximum percentage error of conductivity calculated for the measurement of the sample by the cavity perturbation method. (Assume $\Delta f_0/f_0=2\times10^{-5}$, $\Delta Q_0/Q_0=6\times10^{-2}$, $\varepsilon_r=10$.)

and

$$x=1+\frac{1}{M}\left(\frac{\delta f_0}{f_0}\right)=1+\frac{1}{M}\left(\frac{f_{0s}-f_{0e}}{f_{0e}}\right)$$

$$y=\frac{1}{M}\delta\left(\frac{1}{2Q_0}\right)=\frac{1}{2M}\left(\frac{1}{Q_{0s}}-\frac{1}{Q_{0e}}\right)$$

$$A=\mathrm{Re}\left[\frac{1}{g(N\rho)}\right]$$

$$B=\mathrm{Im}\left[\frac{1}{g(N\rho)}\right].$$

The maximum relative errors $|\Delta\sigma/\sigma|_{\max}$ and $|\Delta\varepsilon/\varepsilon|_{\max}$ of the cavity perturbation method are defined

$$\left|\frac{\Delta\sigma}{\sigma}\right|_{\max}=\left|a_{11}\left(\frac{\Delta f_0}{f_0}\right)\right|+\left|a_{22}\left(\frac{\Delta Q_0}{Q_0}\right)\right|$$
$$\left|\frac{\Delta\varepsilon}{\varepsilon}\right|_{\max}=\left|a_{21}\left(\frac{\Delta f_0}{f_0}\right)\right|+\left|a_{22}\left(\frac{\Delta Q_0}{Q_0}\right)\right| \tag{19}$$

and plotted as functions of radius ($a=0.5$ to 1.5 mm) and conductivity ($\sigma=10^{-4}$ to 1.0) of the sample in Figs. 6 and 7. In these calculations, the measured errors $(\Delta f_0/f_0)=2\times10^{-5}$ and $(\Delta Q_0/Q_0)=6\times10^{-2}$ of the slow scan technique [21] are used. The other conditions ($Q_{0e}=3000$, $f_{0e}=10$ GHz, and $\varepsilon_r=10$) are also assumed.

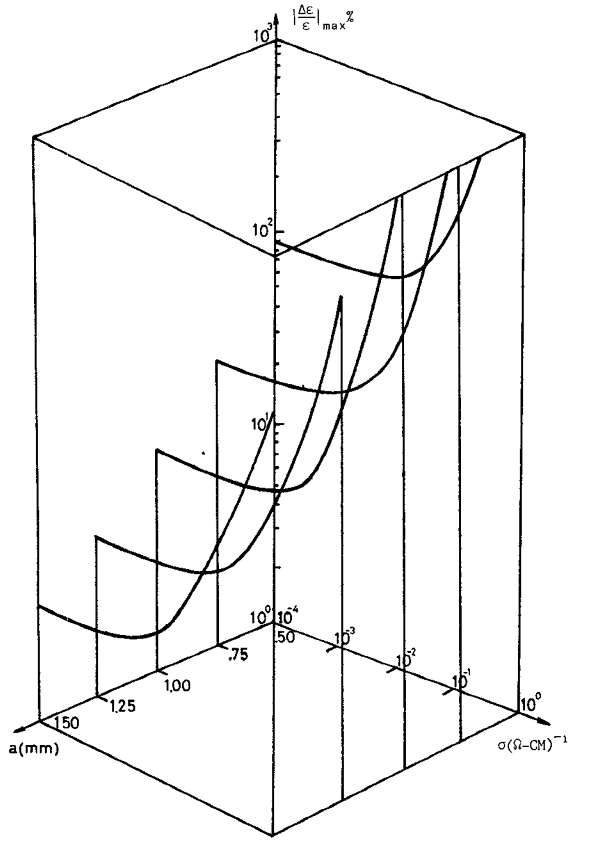

Fig. 7. The maximum percentage error of dielectric constant. (Same as Fig. 6. Assume $\Delta f_0/f_0 = 2\times10^{-5}$, $\Delta Q_0/Q_0 = 6\times10^{-2}$, $\varepsilon_r = 10$.)

From Figs. 6 and 7, the following results are inferred.

1) The maximum error $|\Delta\sigma/\sigma|_{max}$ decreases with increasing conductivity in the low-conductivity region. For $\sigma \leq 10^{-3}$ and $a = 1$-mm samples, the value of $|\Delta\sigma/\sigma|_{max}$ is larger than 100 percent; it decreases to 50 percent for $\sigma = 10^{-2}$ and to 30 percent for $\sigma = 10^{-1}$. Note that $|\Delta\sigma/\sigma|_{max}$ has a minimum value; this minimum value occurs at different conductivities for different radii. After $|\Delta\sigma/\sigma|_{max}$ passes through the minimum value, it begins to increase with increasing conductivity and becomes almost 90 percent for $\sigma = 1.0$ and $a = 1$ mm.

2) The maximum error of $|\Delta\varepsilon/\varepsilon|_{max}$ is constant in the low-conductivity region and approximately equal to 12 percent for $\sigma \leq 10^{-2}$ with $a = 1$ mm. $|\Delta\varepsilon/\varepsilon|_{max}$ also has a very slight minimum value not observable in the figure. After $|\Delta\varepsilon/\varepsilon|_{max}$ passes through this minimum, it begins to increase with increasing conductivity and becomes over 100 percent for the $\sigma = 1.0$ and $a = 1$ mm sample.

IV. Experimental Verification

Samples of silicon having various radii were measured. The primary goal of these measurements was to verify the results of the error analysis of the cavity perturbation method. These measurements employ an iris-coupled reaction-type cavity, constructed from standard WR-90 waveguide operating in the TE_{103} mode given in Fig. 8. A cylindrical sample holder made from Styrofoam is placed at the geometrical center of the inside of the cavity. A

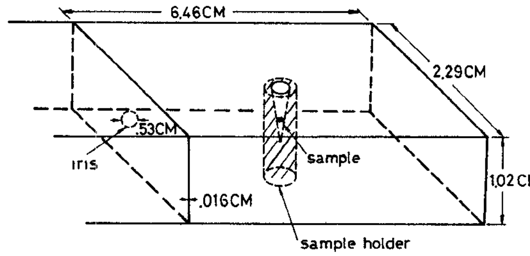

Fig. 8. Iris-coupled TE_{103} rectangular cavity with sample and sample holder.

TABLE I
Conductivity and Dielectric Constant of Silicon ($\sigma_{dc} = 0.018/\Omega\cdot$cm, $\varepsilon_r = 11.85$) as Measured by the Cavity Perturbation Method ($f_{0e} = 9592.8$ MHz, $Q_{0e} = 2630$, $\beta = 0.27$)

a (mm)	f_{os} (MHz)	Q_{ms}	β	Q_{os}	σ $(\Omega\text{-cm})^{-1}$	ε_r	Exp. max. % Uncertainty $\Delta\sigma/\sigma$	$\Delta\varepsilon/\varepsilon$
0.81	9586.1	1850	0.22	2249	0.012	11.54	98	32
1.00	9579.9	1453	0.18	1713	0.023	11.92	56	18
1.50	9549.4	843	0.10	933	0.020	11.21	24	7

small hole drilled on the upper broadside wall of the cavity allows the sample to be inserted into the sample holder without disassembling the cavity and coupling iris.

The measurements of the resonant frequency and Q-factor are performed by using the slow scan technique. The sweep rate of frequency is kept slow enough (about 0.01 MHz/s) to avoid errors in the measurements of the properties of materials due to the uncertainty principle in swept-frequency [22]. The procedure of the determination of σ and ε is described as follows. 1) The resonant frequency f_{0e} and the unloaded Q-factor Q_{0e} of the empty cavity are measured with the sample holder inserted (no sample present). 2) The sample is inserted into the sample holder, and the resonant frequency f_{0s} and unloaded Q-factor Q_{0s} are measured. 3) The values of σ and ε are then computed from the measured values of f_{0e}, Q_{0e}, f_{0s}, and Q_{0s} using (12)–(14).

The results of the measurements of silicon samples are tabulated in Table I. For determining the errors $\Delta\sigma/\sigma$ and $\Delta\varepsilon/\varepsilon$, the dc conductivity ($\sigma_{dc} = 0.018$) and the dielectric constant ($\varepsilon_r = 11.85$) of the silicon samples were measured by the four-probe method and the transmission waveguide method, respectively. The theoretical values of the maximum errors $|\Delta\sigma/\sigma|_{max}$ and $|\Delta\varepsilon/\varepsilon|_{max}$ are then calculated from (19) by assuming that the values of σ and ε provided by the four-probe and the transmission waveguide methods are the "true" values. In Fig. 9, the possible upper and lower bounds of σ and ε as calculated from $|\Delta\sigma/\sigma|_{max}$ and $|\Delta\varepsilon/\varepsilon|_{max}$ are shown by the error bars, and the measured values are marked by the crosses. Those assumed "true" values of σ and ε are also shown by a dashed line in these figures. The results of the measurements of σ and ε show that the theoretical results of the error analysis encompass most of the experimental results.

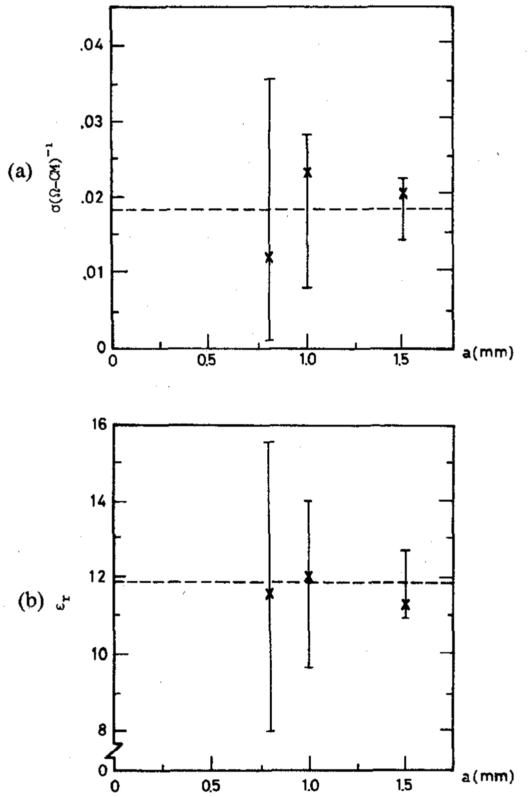

Fig. 9. Theoretical maximum error bounds and experimental results of silicon (σ dc $= 0.018/\Omega\cdot$cm, $\varepsilon_r = 11.85$) of cavity perturbation method. (a) Conductivity and (b) dielectric constant.

V. Summary and Discussion

The resonant frequency shift and Q-factor change of a microwave cavity caused by the insertion of a sample with different σ and ε, and sample sizes have been calculated and shown in the figures. These figures give us a quantity concept of the change of cavity parameters due to the perturbation of samples. An analytical formula for calculating the errors of σ and ε caused by the measured error in the resonant frequency and Q-factor has been derived. From this formula, the order of accuracy of the measurement results of σ and ε by the cavity perturbation method can be predicted. Note that in the error analysis, we assumed that the field applied to the sample was uniform. Hence, the theoretical errors in measured σ and ε decrease with an increasing sample radius. However, when the sample is made larger, the applied field may not be uniform and may introduce error. To account for this, the exact solution (an infinite series) should be used and higher order terms should be taken into consideration.

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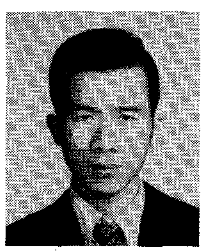

Shuh-Han Chao was born in China on July 29, 1948. He received his Ph.D. degree in electrical engineering from Northwestern University, Evanston, IL in 1981.

Since 1981, he has been working at Materials R & D Center of Chung Shan Institute of Science and Technology, Taiwan, R.O.C. His research activities at present are in the areas of microwave measurement, antenna, optoelectronics, and numerical techniques in electromagnetic theory.